

ASYMPTOTIC BEHAVIOR OF MEROMORPHIC FUNCTIONS WITH EXTREMAL DEFICIENCIES

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Introduction. This paper continues and completes the preceding one of A. Edrei.

I shall adopt the terminology, the bibliographical references and all the notations and conventions of Edrei's paper. Whenever necessary, I shall refer to it as [L]. In view of my frequent use of specific formulae of this paper, as well as of [2], I shall write, for instance, [L, (2.9)] or [2, (2.9)] to denote, respectively, formula (2.9) of [L] or of [2]. Other references will be denoted in the same way as is done in [L].

One of the aims of my investigation is the completion of the proof of Theorem A of [L]. Since the relation [L, (7)] is already proved I have only to examine [L, (8)].

Using Theorem 2 of [L], Edrei had previously proved [L, (8)] for values of μ belonging to the sequence

$$\{1/2 + 1/2q\} \quad (q = 1, 2, \dots).$$

[Notices Amer. Math. Soc. **14** (1967), Abstracts 643–23 (p. 248) and 644–72 (p. 380).]

The methods which I develop here enable me to prove [L, (8)] for all μ in the interval $(\frac{1}{2}, 1)$. They may be summarized as follows:

I. Consider the sets $E_0(r_m)$ and $E_\infty(r_m)$ which appear in Theorem 1 of [L]. The limits of their measures have been determined but it is still possible that these sets be the union of many disjoint intervals. I first show that in some sense each of the sets $E_0(r_m)$ and $E_\infty(r_m)$ is "essentially" an interval.

II. This enables me to return to the distribution of the zeros and poles lying in the annuli

$$(1) \quad R'_m < r = |z| \leq R''_m \quad (R'_m < r_m < R''_m)$$

where R'_m , r_m , and R''_m are quantities satisfying [L, (2.7)]. I prove that almost all the poles in (1) have arguments close to some quantity ω_m and almost all the zeros have arguments close to $\omega_m + \pi$.

III. This knowledge about the zeros and poles of f in (1) is sufficient to determine the asymptotic behavior of $f(z)$ on some circles in the annuli (1).

IV. Theorem 2 of [L] shows that these arguments may be applied to $f'(z)$. The asymptotic evaluation mentioned above, applied to $f'(z)$, indicates that there exists a circle in the annulus (1) such that $f'(z)$ is very small on a single arc $\mathcal{C}_m$ of

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the circumference. By an obvious integration we then verify that $f(z)$ is practically constant on $\mathcal{C}_m$. On the complementary arc $f(z)$ is very large so that $f(z)$ can have only one finite deficient value.

1. Statement of the main results. In addition to the notations of [L] I require the following ones, which will enable me to conveniently refer to some sets which appear in my proofs.

Throughout this paper, I denote by C the set of all the arguments θ such that $-\pi < \theta \leq \pi$.

Since we are only interested in the circular arrangement of the elements of C , the points $\theta = -\pi$ and $\theta = \pi$ will be "identified" and, more generally, all the values $\theta + 2k\pi$ ($k = 0, \pm 1, \pm 2, \pm 3, \dots$) will be considered as different numerical representations of a single element of C .

Beside C , I introduce

I. The sector

$$\mathcal{S}(\omega, \gamma; R', R'') = \{z: \omega - \gamma < \arg z \leq \omega + \gamma; R' < |z| \leq R''\}.$$

II. Put $\theta = \arg z$. The "interval" $\omega - \gamma < \theta \leq \omega + \gamma$, considered as a subset of C , will be denoted by $\Gamma(\omega, \gamma)$.

III. I extend Nevanlinna's notation and denote by $n(\mathcal{D}, f)$ the number of poles of $f(z)$ which fall in the bounded set $\mathcal{D}$. (Multiple poles are counted as often as indicated by their multiplicity.)

With these conventions, we obtain a natural complement to Theorem 1 of [L].

THEOREM 1. *Let $f(z)$ be a meromorphic function of lower order μ ($0 < \mu < 1$) and let*

$$(1.1) \quad \limsup_{r \rightarrow \infty; r \notin \mathcal{E}} \frac{N(r, 1/f)}{T(r, f)} = u, \quad \limsup_{r \rightarrow \infty; r \notin \mathcal{E}} \frac{N(r, f)}{T(r, f)} = v,$$

where $\mathcal{E}$ is any fixed set of density zero.

Assume that u and v satisfy

$$(1.2) \quad u < 1, \quad v < 1$$

and

$$(1.3) \quad \sin^2 \pi\mu = u^2 + v^2 - 2uv \cos \pi\mu.$$

Then, with every sequence $\{r_m\}$ of Pólya peaks of order μ of $T(r, f)$, it is possible to associate four sequences $\{\omega_m\}$, $\{\eta_m\}$, $\{\rho'_m\}$, and $\{\rho''_m\}$ having the following properties:

$$(1.4) \quad 0 < \eta_m < \pi \quad (m = 1, 2, \dots), \quad \eta_m \rightarrow \pi \quad \text{as } m \rightarrow \infty,$$

$$(1.5) \quad \rho'_m \rightarrow +\infty, \quad r_m/\rho'_m \rightarrow +\infty, \quad \rho''_m/r_m \rightarrow +\infty \quad \text{as } m \rightarrow \infty,$$

and

$$(1.6) \quad \begin{aligned} n(\mathcal{S}(\omega_m, \eta_m; \rho'_m, \rho''_m), 1/f) &= o(T(r_m, f)), \\ n(\mathcal{S}(\omega_m + \pi, \eta_m; \rho'_m, \rho''_m), f) &= o(T(r_m, f)), \end{aligned}$$

as $m \rightarrow \infty$.

Theorem 1 enables us to obtain an asymptotic evaluation of $f(z)$ which leads to

THEOREM 2. *Let the assumptions and notations of Theorem 1 be unchanged, and let $s(0)$ and $s(\infty)$ be the quantities defined by [L, (2.4)] and [L, (2.5)], and let ε ($0 < \varepsilon < \frac{1}{2} \min \{\sigma(\infty), \sigma(0)\}$) be given.*

Then there exists a sequence $\{\omega_m\}$, a positive sequence $\{\sigma_m\}$ ($\sigma_m \rightarrow +\infty$) and a constant $K > 0$, such that

$$(1.7) \quad \begin{aligned} \log |f(re^{i\theta})| &> KT(r, f) & (\theta \in \Gamma(\omega_m, s(\infty)/2 - \varepsilon)), \\ \log |f(re^{i\theta})| &< -KT(r, f) & (\theta \in \Gamma(\omega_m + \pi, s(0)/2 - \varepsilon)), \end{aligned}$$

provided

- (i) $r \rightarrow +\infty$ in the intervals $\sigma_m^{-1}r_m < r \leq \sigma_m r_m$;
- (ii) r avoids in each of these intervals an exceptional set $\mathcal{E}_m$ of measure not greater than $\sigma_m^{-2}r_m$.

From this theorem we deduce at once that:

The values of r for which the inequalities (1.7) are valid have upper density one.

Theorem 2 and the well-known relations between a function and its derivative lead to

THEOREM 3. *Let $f(z)$ be a meromorphic function satisfying the conditions of Theorem 1.*

Then, if $F'(z) = f(z)$, and if $F(z)$ is meromorphic, it has at most two deficient values.

Theorem 3 is not vacuous because the meromorphic function

$$F(z) = \frac{\prod_{j=1}^{\infty} (1 + zn^{-1/\mu})}{\prod_{j=1}^{\infty} (1 - zn^{-(\mu+1)/\mu})} \quad (\tfrac{1}{2} < \mu < 1)$$

satisfies the conditions $\delta(0, F) = 1 - \sin \pi\mu$, $\delta(\infty, F) = 1$ [see for example R. Nevanlinna, *Eindeutige analytische Funktionen*, p. 232], and hence, in view of Theorem 2 of [L], $u(F') = \sin \pi\mu$, $v(F') = 0$.

This shows that $f = F'$ satisfies the conditions of Theorem 1.

It might be of interest to investigate whether there exist functions $f(z)$, satisfying the conditions of Theorem 1 with $0 < \delta(\infty, f) < 1$, and having a meromorphic integral. I am at present unable to answer this question.

As an immediate consequence of Theorem 3 and [L, Theorem 2] we now obtain [L, (8)] which I restate for completeness.

If $f(z)$ is a meromorphic function of lower order μ ($\frac{1}{2} < \mu < 1$), if $\delta(\infty, f) = 1$, and if $\Delta(f) = 2 - \sin \pi\mu$, then $\nu(f) = 2$.

Hence any function $f(z)$ satisfying the above conditions has precisely one finite deficient value τ , such that $\delta(\tau, f) = 1 - \sin \pi\mu$, and $f(z) - \tau$ has the asymptotic behavior described in Theorem 2.

2. **Structure of the sets $E_0(r_m)$ and $E_\infty(r_m)$.** Let E denote a measurable subset of C and let

$$(2.1) \quad \gamma = \frac{1}{2} \text{meas } E.$$

Consider the function $\mathcal{M}(\omega) = \text{meas } \{E - \Gamma(\omega, \gamma)\}$ which is clearly a nonnegative function of ω , defined and continuous on C . Let $\bar{\omega}$ be any one of the values of ω such that

$$\mathcal{M}(\bar{\omega}) = \inf_{\omega \in C} \mathcal{M}(\omega) = \chi.$$

We shall say that $\bar{\omega}$ is a center of E .

The inequalities $0 \leq \mathcal{M}(\omega) \leq 2\gamma$, $\mathcal{M}(\omega) \leq 2(\pi - \gamma)$, are obvious.

If $\gamma = 0$ or $\gamma = \pi$, we have $\mathcal{M}(\omega) \equiv 0$ and $\chi = 0$ (trivially); in both cases, every $\omega \in C$ is a center of E . If $0 < \gamma < \pi$, the inequality $\chi > 0$ is possible; the quantity χ then represents, in some sense, the total measure of the "gaps" in E .

If $\gamma > 0$ and $\chi = 0$, we may think of E as being, apart from a set of zero measure, an interval on C . The following lemma shows that, for functions satisfying (1.3), the sets $E_\infty(r_m)$ and $E_0(r_m)$, tend, as $m \rightarrow +\infty$, toward this "single interval" structure.

LEMMA 1. *Let $f(z)$ satisfy the hypotheses of Theorem 1, and let $\{r_m\}$ be a sequence of Pólya peaks of order μ of $T(r, f)$. Let*

$$(2.2) \quad \gamma_m = \frac{1}{2} \text{meas } E_\infty(r_m).$$

Then, there exists a sequence $\{\omega_m\}$ such that

$$(2.3) \quad \begin{aligned} \lim_{m \rightarrow \infty} \text{meas } \{E_\infty(r_m) - \Gamma(\omega_m, \gamma_m)\} &= 0, \\ \lim_{m \rightarrow \infty} \text{meas } \{E_0(r_m) - \Gamma(\omega_m + \pi, \pi - \gamma_m)\} &= 0. \end{aligned}$$

Before proving Lemma 1 we prove two elementary lemmas.

LEMMA 2. *Let E be a measurable subset of C . Then, if w is any value, real or complex,*

$$\frac{1}{2\pi} \int_E |\log |1 - we^{i\theta}| | d\theta \leq \left\{ \log(1 + |w|) + \left(1 + \log^+ \frac{1}{\text{meas } E}\right) \right\} \text{meas } E.$$

Proof. Put $\arg w + \theta = \phi$.

Then

$$(2.4) \quad |1 - we^{i\theta}| = |1 - |w|e^{i\phi}| = |e^{-i\phi} - |w|| \geq |\sin \phi|,$$

which may be sharpened to

$$(2.5) \quad |1 - we^{i\theta}| \geq 1,$$

if $\pi/2 \leq |\phi| \leq \pi$.

Now

$$(2.6) \quad |\sin \phi| \geq (2/\pi)|\phi| \quad (|\phi| \leq \pi/2)$$

and hence, if $\text{meas } E = \mathcal{K}$ and

$$(2.7) \quad \alpha = \min \{\pi, \mathcal{K}\},$$

we observe, with Edrei and Fuchs [7, p. 338], that (2.4), (2.5), (2.6), and (2.7) imply

$$(2.8) \quad \begin{aligned} I(E) &= \frac{1}{2\pi} \int_E \log^+ \left| \frac{1}{1 - we^{i\theta}} \right| d\theta \leq \frac{1}{\pi} \int_0^{\alpha/2} \log \left(\frac{1}{\sin \phi} \right) d\phi \\ &\leq \frac{1}{\pi} \int_0^{\alpha/2} \log \left(\frac{\pi}{2\phi} \right) d\phi \leq - \int_0^{\alpha/2\pi} \log t \, dt. \end{aligned}$$

By definition $\alpha/2\pi \leq \mathcal{K}/2\pi \leq 1$, so that (2.8) yields

$$(2.9) \quad I(E) \leq - \int_0^{\mathcal{K}/2\pi} \log t \, dt = \frac{\mathcal{K}}{2\pi} + \frac{\mathcal{K}}{2\pi} \log \left(\frac{2\pi}{\mathcal{K}} \right) \leq \mathcal{K} \left(1 + \log^+ \frac{1}{\mathcal{K}} \right).$$

We now obtain Lemma 2 by integrating over E the obvious relation

$$|\log |1 - we^{i\theta}| | \leq \log (1 + |w|) + \log^+ |1/(1 - we^{i\theta})|,$$

and using the estimate (2.9).

LEMMA 3. *Let E be a measurable subset of C , and let*

$$(2.10) \quad \text{meas } E = 2\gamma.$$

Assume

$$(2.11) \quad \text{meas } \{E - \Gamma(0, \gamma)\} \geq 2\xi.$$

Then, if t is restricted to the range

$$(2.12) \quad \sigma^{-1} \leq t \leq \sigma \quad (1 < \sigma < +\infty),$$

we have

$$(2.13) \quad \frac{1}{2\pi} \int_{-\gamma}^{\gamma} \log |1 + te^{i\theta}| \, d\theta - \frac{1}{2\pi} \int_E \log |1 + te^{i\theta}| \, d\theta \geq K = K(\sigma, \xi).$$

The constant K which appears in (2.13) may be chosen equal to

$$(2.14) \quad K(\sigma, \xi) = 2\xi \sin^2 (\xi/2)/\pi\{4 + \sigma(1 + \sigma^2)\},$$

which is clearly positive for $0 < \xi \leq \pi/2$.

Proof. Put

$$(2.15) \quad \{E - \Gamma(0, \gamma)\} = G_1, \quad \{E \cap \Gamma(0, \gamma)\} = G_2,$$

so that $E = \{G_1 \cup G_2\}$, $\{G_1 \cap G_2\} = 0$. Hence, in view of (2.10) and (2.11), we have

$$(2.16) \quad 0 \leq 2\xi \leq 2\eta = \text{meas } G_1 \leq 2(\pi - \gamma), \quad \text{meas } G_2 = 2(\gamma - \eta).$$

We now use the familiar remark that, for any fixed $t > 0$, $\log |1 + te^{i\theta}|$ is an even function of θ , strictly decreasing as θ varies from 0 to π . By (2.15) and (2.16); this leads to the obvious inequalities:

$$\int_{G_2} \log |1 + te^{i\theta}| d\theta \leq 2 \int_0^{\gamma-\eta} \log |1 + te^{i\theta}| d\theta,$$

and

$$\int_{G_1} \log |1 + te^{i\theta}| d\theta \leq 2 \int_{\gamma}^{\gamma+\eta} \log |1 + te^{i\theta}| d\theta = 2 \int_{\gamma-\eta}^{\gamma} \log |1 + te^{i(\phi+\eta)}| d\phi,$$

which, when added, yield

$$(2.17) \quad \int_E \log |1 + te^{i\theta}| d\theta \leq 2 \int_0^{\gamma} \log |1 + te^{i\theta}| d\theta - 2 \int_{\gamma-\eta}^{\gamma} \log \left| \frac{1 + te^{i\theta}}{1 + te^{i(\theta+\eta)}} \right| d\theta.$$

Consider now the positive function

$$(2.18) \quad H(t, \theta, \eta) = \left| \frac{1 + te^{i\theta}}{1 + te^{i(\theta+\eta)}} \right|^2 = 1 + \frac{2t\{\cos \theta - \cos(\theta+\eta)\}}{1 + t^2 + 2t \cos(\theta+\eta)},$$

which appears in the last integral of (2.17). From (2.16) we deduce $\gamma + \eta/2 \leq \pi - \eta/2$, $\eta/2 \leq \gamma - \eta/2$, and hence

$$(2.19) \quad \cos \theta - \cos(\theta + \eta) = 2 \sin(\theta + \eta/2) \sin \eta/2 \geq 2 \sin^2(\eta/2) \quad (\gamma - \eta \leq \theta \leq \gamma).$$

Combining (2.18), (2.12) and (2.19), we find

$$\begin{aligned} H(t, \theta, \eta) - 1 &\geq \frac{4\sigma^{-1} \sin^2(\eta/2)}{(1 + \sigma)^2} = \zeta, \\ \log H(t, \theta, \eta) &\geq \log(1 + \zeta) > \frac{\zeta}{1 + \zeta} \geq \frac{4 \sin^2(\eta/2)}{4 + \sigma(1 + \sigma)^2}, \\ \int_{\gamma-\eta}^{\gamma} \log H(t, \theta, \eta) d\theta &\geq \frac{4\eta \sin^2(\eta/2)}{4 + \sigma(1 + \sigma)^2}, \end{aligned}$$

and, since $0 < \xi \leq \eta$, it is obvious that (2.17), (2.18) and (2.19) imply (2.13) and (2.14). This completes the proof of Lemma 3.

Proof of Lemma 1. The assumptions of Lemma 1 coincide with those of Theorem 1 so that (1.1), (1.2) and (1.3) hold. Then, in view of assertion II of [L, Theorem 1], we have

$$(2.20) \quad 0 < \beta = \lim_{m \rightarrow \infty} \gamma_m = \frac{1}{\mu} \cos^{-1} v = \frac{s(\infty)}{2} < \pi \quad \left(0 < \cos^{-1} v \leq \frac{\pi}{2} \right),$$

where $2\gamma_m = \text{meas } E_{\infty}(r_m)$.

Let ω_m be a center of $E_{\infty}(r_m)$; we first examine the implications of

$$(2.21) \quad \limsup_{m \rightarrow \infty} \text{meas } \{E_{\infty}(r_m) - \Gamma(\omega_m, \gamma_m)\} \neq 0.$$

From (2.21) we deduce the existence of a constant $\xi > 0$ and of an unbounded sequence $\mathcal{M}$, of positive integers, such that

$$(2.22) \quad \text{meas} \{E_\infty(r_m) - \Gamma(\omega_m, \gamma_m)\} \geq 2\xi \quad (m \in \mathcal{M}).$$

Let $\sigma > 1$ be a given, fixed quantity and let $a = |a|e^{i\psi}$ be any one of the zeros of $f(z)$ such that

$$(2.23) \quad \sigma^{-1}r_m < |a| \leq \sigma r_m.$$

In view of the extremal character of the centers ω_m , the inequalities (2.22) remain true if ω_m is replaced by any other point of C ; in particular

$$(2.24) \quad \text{meas} \{E_\infty(r_m) - \Gamma(\psi + \pi, \gamma_m)\} \geq 2\xi \quad (m \in \mathcal{M}).$$

The transformation of the set C defined by

$$(2.25) \quad \phi = \theta - \psi - \pi \quad (\theta \in C),$$

is a "translation" which leaves C invariant and transforms the subsets of C without affecting their measures. In particular, the sets $E_\infty(r_m)$, $\Gamma(\psi + \pi, \gamma_m)$ are transformed, respectively, into sets $\tilde{E}_m$ and $\Gamma(0, \gamma_m)$ and the inequalities (2.24) become

$$\text{meas} \{\tilde{E}_m - \Gamma(0, \gamma_m)\} \geq 2\xi \quad (m \in \mathcal{M}).$$

Hence, in view of (2.23) and (2.25), Lemma 3 yields

$$(2.26) \quad \begin{aligned} \frac{1}{2\pi} \int_{E_\infty(r_m)} \log \left| 1 - \frac{r_m e^{i\theta}}{a} \right| d\theta &= \frac{1}{2\pi} \int_{\tilde{E}_m} \log \left| 1 + \frac{r_m}{|a|} e^{i\phi} \right| d\phi \\ &\leq -K + \frac{1}{\pi} \int_0^{\gamma_m} \log \left| 1 + \frac{r_m e^{i\theta}}{|a|} \right| d\theta \\ &\quad (m \in \mathcal{M}; K = K(\sigma, \xi)), \end{aligned}$$

where the positive constant K depends on no parameters other than σ and ξ .

There are $\tilde{n}_m = n(\sigma r_m, 1/f) - n(\sigma^{-1}r_m, 1/f)$ zeros of $f(z)$ characterized by the inequalities (2.23). Since our assumptions imply the validity of assertion II of [L, Theorem 1], we deduce from [L, (2.9)] and [L, (2.11)]

$$(2.27) \quad \lim_{m \rightarrow \infty; m \in \mathcal{M}} \frac{\tilde{n}_m}{T(r_m, f)} = \mu u(\sigma^\mu - \sigma^{-\mu}).$$

We denote by a_j the zeros of $f(z)$ and by b_j its poles and, in the following inequality, confine our attention, and our summations, to the zeros satisfying (2.23). Then (2.26) and (2.27) yield

$$(2.28) \quad \begin{aligned} \sum \frac{1}{2\pi} \int_{E_\infty(r_m)} \log \left| 1 - \frac{r_m e^{i\theta}}{a_j} \right| d\theta \\ \leq \frac{1}{\pi} \int_0^{\gamma_m} \left\{ \sum \log \left| 1 + \frac{r_m e^{i\theta}}{|a_j|} \right| \right\} d\theta - K_1 T(r_m, f) u(1 + o(1)) \\ (m \rightarrow +\infty, m \in \mathcal{M}), \end{aligned}$$

where $K_1 = K(\sigma, \xi)\mu(\sigma^\mu - \sigma^{-\mu})$.

We now examine a proof of Edrei [2, pp. 87–94] and consider, in particular, the fundamental inequality [2, (2.18)]. With our notations this inequality implies

$$\begin{aligned}
 m(r_m, f) \leq & \sum_{0 < |a_j| \leq R_m} \frac{1}{2\pi} \int_{E_\infty(r_m)} \log \left| 1 - \frac{r_m e^{i\theta}}{a_j} \right| d\theta \\
 & - \sum_{0 < |b_j| \leq R_m} \frac{1}{2\pi} \int_{E_\infty(r_m)} \log \left| 1 - \frac{r_m e^{i\theta}}{b_j} \right| d\theta \\
 & + 15 \frac{r_m}{R_m} T(2R_m) + o(T(r_m)) \\
 & (m \rightarrow \infty, r_m \leq \frac{1}{2}R_m, T(r) = T(r, f)),
 \end{aligned}
 \tag{2.29}$$

where it is understood that, subject to the restriction $R_m \geq 2r_m$, the size of the error term is not affected by the choice of R_m .

The arguments in [2, p. 90] may be repeated with the following minor modification: instead of using [2, (2.20)] to estimate all the terms of the first sum in the right-hand side of (2.29), we use (2.28) to evaluate the contribution of all the a_j such that $\sigma^{-1}r_m < |a_j| \leq \sigma r_m$.

We thus obtain

$$\begin{aligned}
 T(r_m) \leq & \frac{1}{\pi} \int_0^{\gamma_m} \left\{ \sum_{0 < |a_j| \leq R_m} \log \left| 1 + \frac{r_m e^{i\theta}}{|a_j|} \right| \right\} d\theta + \int_0^{\pi - \gamma_m} \left\{ \sum_{0 < |b_j| \leq R_m} \log \left| 1 + \frac{r_m e^{i\theta}}{|b_j|} \right| \right\} d\theta \\
 & + 15 \frac{r_m}{R_m} T(2R_m) + o(T(r_m)) - K_1 u T(r_m) \quad (m \rightarrow +\infty, m \in \mathcal{M}, r_m \leq \frac{1}{2}R_m),
 \end{aligned}$$

instead of [2, (2.22)].

In view of (2.20), $0 < \gamma_m < \pi$ ($m > m_0$) and we obtain, as in [2],

$$\begin{aligned}
 T(r_m) \leq & \int_0^{R_m} N_0(t) P(t, r_m, \gamma_m) dt + \int_0^{R_m} N_\infty(t) P(t, r_m, \pi - \gamma_m) dt - K_1 u T(r_m) \\
 & + A \frac{r_m}{R_m} T(2R_m) + o(T(r_m)) \quad (m \rightarrow +\infty, m \in \mathcal{M}),
 \end{aligned}
 \tag{2.30}$$

where A (> 0) is an absolute constant and the symbols N_0 , N_∞ , P have the same meaning as in [2] or in [L, (4.8)].

The main difference between (2.30) and [L, (4.8)] is the presence, in (2.30), of the negative quantity $-K_1 u T(r_m)$. The arguments which, in [L], lead to [L, (4.12)] now yield

$$\sin \pi \mu \leq u \sin \beta \mu + v \sin (\pi - \beta) \mu - K_1 u \sin \pi \mu.
 \tag{2.31}$$

Using the Cauchy-Schwarz inequality, as in [L, (5.1)], we deduce from (2.31)

$$(K_1 u + 1) \sin^2 \pi \mu \leq u^2 + v^2 - 2uv \cos \pi \mu,$$

and hence, by (1.3), $u = 0$.

We have thus shown that, if $u \neq 0$, the relation (2.21) is impossible and therefore

$$(2.32) \quad \lim_{m \rightarrow \infty} \text{meas} \{E_{\infty}(r_m) - \Gamma(\omega_m, \gamma_m)\} = 0.$$

This is the first of the relations (2.3).

From (2.32) and (2.20) we deduce

$$(2.33) \quad \lim_{m \rightarrow \infty} \text{meas} \{E_{\infty}(r_m) \cap \Gamma(\omega_m, \gamma_m)\} = \lim_{m \rightarrow \infty} (2\gamma_m) = s(\infty).$$

The sets $E_0(r_m)$ and $E_{\infty}(r_m)$ are disjoint by definition so that C may be represented as the union of three pairwise disjoint sets:

$$(2.34) \quad C = \{E_0(r_m) \cup E_{\infty}(r_m) \cup E_1(r_m)\}.$$

Then, by [L, (2.6)]

$$(2.35) \quad \lim_{m \rightarrow \infty} \text{meas} E_1(r_m) = 0.$$

From (2.34) we deduce

$$\begin{aligned} \text{meas} \{E_0(r_m) \cap \Gamma(\omega_m, \gamma_m)\} + \text{meas} \{E_{\infty}(r_m) \cap \Gamma(\omega_m, \gamma_m)\} \\ + \text{meas} \{E_1(r_m) \cap \Gamma(\omega_m, \gamma_m)\} = 2\gamma_m, \end{aligned}$$

and hence, by (2.33) and (2.35)

$$(2.36) \quad \lim_{m \rightarrow \infty} \text{meas} \{E_0(r_m) \cap \Gamma(\omega_m, \gamma_m)\} = 0.$$

Finally, $\Gamma(\omega_m, \gamma_m)$ and $\{(\pi + \omega_m, \pi - \gamma_m)\}$ are disjoint and their union is C . Therefore

$$\text{meas} \{E_0(r_m) \cap \Gamma(\pi + \omega_m, \pi - \gamma_m)\} + \text{meas} \{E_0(r_m) \cap \Gamma(\omega_m, \gamma_m)\} = \text{meas} E_0(r_m)$$

which, in view of (2.36), yields

$$\lim_{m \rightarrow \infty} \text{meas} \{E_0(r_m) \cap \Gamma(\pi + \omega_m, \pi - \gamma_m)\} = \lim_{m \rightarrow \infty} \text{meas} E_0(r_m),$$

and proves the second relation in (2.3).

We have thus completed the proof of Lemma 1 in the case $u \neq 0$. If $u = 0$, we certainly have $v \neq 0$ (by (1.3)) and hence Lemma 1 follows from the consideration of the function $1/f$, instead of f .

3. Arguments of the zeros and poles of $f(z)$. Lemma 1 gives a precise meaning to step I of the general argument outlined in the Introduction. The following Lemma 4 clarifies, in a similar manner, step II.

LEMMA 4. *Let the assumptions and notations of Lemma 1 be unchanged. Let σ and η be given, fixed quantities such that*

$$(3.1) \quad 1 < \sigma, \quad 0 < \pi - \eta < \min(s(0), s(\infty)).$$

[$s(0)$ and $s(\infty)$ are defined as in [L, Theorem 1]].

I. Then, if $K(\sigma, \xi)$ is the constant in (2.14), and if

$$(3.2) \quad 2K_2 = K(\sigma, (\pi - \eta)/2),$$

we have

$$(3.3) \quad \frac{1}{2\pi} \int_{E_\infty(r_m)} \log \left| 1 - \frac{r_m e^{i\theta}}{a} \right| d\theta \leq \frac{1}{\pi} \int_0^{\gamma_m} \log \left| 1 + \frac{r_m e^{i\theta}}{|a|} \right| d\theta - K_2,$$

provided

$$(3.4) \quad a \in \mathcal{S}(\omega_m, \eta; \sigma^{-1}r_m, \sigma r_m)$$

and $m > m_2$.

The bound m_2 , which depends on $f, \{r_m\}, \sigma$ and η , holds uniformly for all a satisfying (3.4).

II. The counting functions of all zeros and poles of $f(z)$ satisfy the relations

$$(3.5) \quad n(\mathcal{S}(\omega_m, \eta; \sigma^{-1}r_m, \sigma r_m), 1/f) = o(T(r_m, f)),$$

$$(3.6) \quad n(\mathcal{S}(\omega_m + \pi, \eta; \sigma^{-1}r_m, \sigma r_m), f) = o(T(r_m, f)),$$

as $m \rightarrow \infty$.

Proof. The assumptions of Lemma 4 coincide with those of Lemma 1 as well as with those of [L, Theorem 1, assertion II]. Hence if $\{r_m\}$ is a sequence of Pólya peaks, of order μ , of $T(r) = T(r, f)$, we see that (2.2) and (2.3) hold and that

$$(3.7) \quad 0 < \lim_{m \rightarrow \infty} 2\gamma_m = s(\infty) < 2\pi, \quad 0 < \lim_{m \rightarrow \infty} 2(\pi - \gamma_m) = s(0) < 2\pi.$$

Consider the sets

$$(3.8) \quad \begin{aligned} C_{m1} &= \{E_\infty(r_m) \cap \Gamma(\omega_m, \gamma_m)\}, \\ C_{m2} &= \{E_\infty(r_m) - \Gamma(\omega_m, \gamma_m)\} = \{E_\infty(r_m) - C_{m1}\}, \\ C_{m3} &= \{\Gamma(\omega_m, \gamma_m) - E_\infty(r_m)\} = \{\Gamma(\omega_m, \gamma_m) - C_{m1}\}, \end{aligned}$$

and notice that, in view of the first of the relations (2.3), we have

$$(3.9) \quad \lim_{m \rightarrow \infty} \text{meas } C_{m1} = \lim_{m \rightarrow \infty} \text{meas } E_\infty(r_m) = s(\infty),$$

and hence, as $m \rightarrow \infty$,

$$(3.10) \quad (\text{meas } C_{m2} + \text{meas } C_{m3}) = c_m \rightarrow 0.$$

Let $\int$ indicate integration of some measurable function defined on C ; then, by (3.8),

$$(3.11) \quad \int_{E_\infty(r_m)} - \int_{\Gamma(\omega_m, \gamma_m)} = \int_{C_{m2}} - \int_{C_{m3}} = \zeta_m.$$

In particular, if we apply (3.11) to the function

$$(3.12) \quad \log |1 - r_m e^{i\theta}/a| \quad (|a| \leq \sigma r_m),$$

we obtain, in view of (3.10) and Lemma 2,

$$(3.13) \quad |\zeta_m| \leq 4\pi c_m \{\log(1+\sigma) + 1 + \log(1/c_m)\} \quad (m > m_0(\sigma)).$$

Now let

$$(3.14) \quad a = |a|e^{i\psi}$$

satisfy the condition (3.4) so that

$$(3.15) \quad \sigma^{-1} \leq r_m/|a| < \sigma,$$

$$(3.16) \quad \psi = \omega_m + \kappa\eta \quad (-1 < \kappa \leq 1).$$

With a suitable choice of $m_1(\sigma, \eta)$ we may, in view of (3.1), assume

$$(3.17) \quad 0 < \pi - \eta < \min(2\gamma_m, 2\pi - 2\gamma_m) \quad (m > m_1(\sigma, \eta)).$$

The change of variable $\phi = \theta - \pi - \psi$ leads to

$$(3.18) \quad \int_{\Gamma(\omega_m, \gamma_m)} \log \left| 1 - \frac{r_m e^{i\theta}}{a} \right| d\theta = \int_{\Gamma(\tilde{\omega}_m, \gamma_m)} \log \left| 1 + \frac{r_m e^{i\phi}}{|a|} \right| d\phi,$$

where

$$(3.19) \quad \tilde{\omega}_m = \omega_m - \psi - \pi.$$

Using (3.16) in (3.19), we find $\tilde{\omega}_m = -\pi - \kappa\eta$, and therefore

$$\begin{aligned} -\pi < \tilde{\omega}_m < -\pi + \eta & \text{ if } -1 < \kappa < 0, \\ \pi - \eta \leq 2\pi + \tilde{\omega}_m \leq \pi & \text{ if } 0 \leq \kappa \leq 1. \end{aligned}$$

Hence if ω'_m is defined by the relations

$$\begin{aligned} \omega'_m &= \tilde{\omega}_m & \text{if } -1 < \kappa < 0, \\ \omega'_m &= 2\pi + \tilde{\omega}_m & \text{if } 0 \leq \kappa \leq 1, \end{aligned}$$

we always have

$$(3.20) \quad \Gamma(\tilde{\omega}_m, \gamma_m) = \Gamma(\omega'_m, \gamma_m),$$

with

$$(3.21) \quad 0 < \pi - \eta \leq |\omega'_m| \leq \pi.$$

Before applying Lemma 3 to the last integral in (3.18), we require the following elementary remark:

$$(3.22) \quad \text{meas} \{ \Gamma(\omega'_m, \gamma_m) - \Gamma(0, \gamma_m) \} = \min \{ |\omega'_m|, 2\gamma_m, 2(\pi - \gamma_m) \} \\ (|\omega'_m| \leq \pi, 0 < \gamma_m < \pi).$$

In order to verify this relation consider the set

$$(3.23) \quad \mathcal{G}(\omega) = \{\Gamma(\omega, \gamma) - \Gamma(0, \gamma)\},$$

for γ fixed and ω variable.

A. First assume $0 < 2\gamma \leq \pi$ and, as ω increases from 0 to π , follows the positions of the points $\omega - \gamma$, $\omega + \gamma$ in the interval $(-\gamma, 2\pi - \gamma]$. Obviously

$$(3.24) \quad \begin{aligned} \mathcal{G}(\omega) &= (\gamma, \omega + \gamma] & \text{if } 0 < \omega < 2\gamma, \\ \mathcal{G}(\omega) &= (\omega - \gamma, \omega + \gamma] & \text{if } 2\gamma < \omega \leq \pi. \end{aligned}$$

B. Similarly, if $\pi < 2\gamma < 2\pi$, then

$$(3.25) \quad \begin{aligned} \mathcal{G}(\omega) &= (\gamma, \omega + \gamma] & \text{if } 0 < \omega \leq 2(\pi - \gamma), \\ \mathcal{G}(\omega) &= (\gamma, 2\pi - \gamma) & \text{if } 2(\pi - \gamma) < \omega \leq \pi. \end{aligned}$$

The relations (3.23), (3.24) and (3.25) yield

$$\text{meas } \mathcal{G}(\omega) = \text{meas } \mathcal{G}(-\omega) = \min(|\omega|, 2\gamma, 2(\pi - \gamma)) \quad (|\omega| \leq \pi, 0 < \gamma \leq \pi),$$

and (3.22) follows.

In view of (3.17) and (3.21) we have

$$0 < \pi - \eta \leq \min(|\omega'_m|, 2\gamma_m, 2\pi - 2\gamma_m) \quad (m > m_1(\sigma, \eta)),$$

which used in (3.22) yields

$$(3.26) \quad \text{meas } \{\Gamma(\omega'_m, \gamma_m) - \Gamma(0, \gamma_m)\} \geq \pi - \eta > 0.$$

Consider the constant $K(\sigma, \xi)$ defined by (2.14) and set

$$2K_2 = 2K_2(\sigma, \eta) = K(\sigma, \tfrac{1}{2}(\pi - \eta)) > 0.$$

By (3.15), (3.26) and Lemma 3, we find

$$(3.27) \quad \frac{1}{2\pi} \int_{\Gamma(\omega'_m, \gamma_m)} \log \left| 1 + \frac{r_m e^{i\theta}}{|a|} \right| d\theta \leq -2K_2 + \frac{1}{\pi} \int_0^{\gamma_m} \log \left| 1 + \frac{r_m e^{i\theta}}{|a|} \right| d\theta$$

($m > m_1(\sigma, \eta)$).

In view of (3.18) and (3.20) the left-hand side of (3.27) may be replaced by

$$\frac{1}{2\pi} \int_{\Gamma(\omega'_m, \gamma_m)} \log \left| 1 - \frac{r_m e^{i\theta}}{a} \right| d\theta.$$

If we combine the resulting inequality with (3.11), we find

$$(3.28) \quad \frac{1}{2\pi} \int_{E_\infty(r_m)} \log \left| 1 - \frac{r_m e^{i\theta}}{a} \right| d\theta \leq -2K_2 + \frac{\zeta_m}{2\pi} + \frac{1}{\pi} \int_0^{\gamma_m} \log \left| 1 + \frac{r_m e^{i\theta}}{|a|} \right| d\theta$$

($m > m_1(\sigma, \eta)$).

Now by (3.10) and (3.13)

$$\lim_{m \rightarrow \infty} \zeta_m = 0,$$

uniformly for all a satisfying (3.4). Hence, if m_2 is chosen large enough, the inequality $m > m_2$ implies $m > m_1(\sigma, \eta)$, $\zeta_m/2\pi \leq K_2$, and (3.3) follows from (3.28). We have thus proved assertion I of Lemma 4.

Proof of assertion II of Lemma 4. The parameters σ, η , as well as the sequence $\{\omega_m\}$, are fixed. Explicit reference to all these quantities is unnecessary and we simplify our notation by setting

$$\tilde{n}_m = n(\mathcal{S}(\omega_m, \eta; \sigma^{-1}r_m, \sigma r_m), 1/f).$$

Assume that (3.5) is false. Then there exists some constant $\xi > 0$ and some unbounded sequence $\mathcal{M}$, of positive integers such that

$$\tilde{n}_m > \xi T(r_m) \quad (m \in \mathcal{M}).$$

This yields a contradiction as may be seen by a repetition, with minor modifications, of the proof of Lemma 1:

(i) start from (2.29). Consider its right-hand side and use (3.3) (instead of (2.28)) to estimate the contribution of the $\tilde{n}_m$ terms involving the zeros of $f(z)$ in

$$\mathcal{S}(\omega_m, \eta; \sigma^{-1}r_m, \sigma r_m);$$

(ii) we are thus led to an inequality such as (2.30) with $-K_1 u T(r_m)$ replaced by $-K_2 \xi T(r_m)$, and finally to

$$\sin \pi \mu \leq u \sin \beta \mu + v \sin (\pi - \beta) \mu - K_2 \xi \sin \pi \mu$$

(instead of (2.31)).

Hence $\xi = 0$, a contradiction which proves (3.5). The relation (3.6) is obtained by applying our arguments to $1/f$ instead of f . This completes the proof of Lemma 4.

4. Proof of Theorem 1. Let $l > 2$ be a fixed integer. By Lemma 4 it is possible to determine m_l so that $m > m_l$ implies

$$n(\mathcal{S}(\omega_m, \pi - 1/l; r_m/l, lr_m), 1/f) + n(\mathcal{S}(\omega_m + \pi, \pi - 1/l; r_m/l, lr_m), f) < T(r_m)/l.$$

We then set

$$(4.1) \quad \eta_m = \pi - 1/l, \quad \rho'_m = r_m/l, \quad \rho''_m = lr_m$$

for

$$(4.2) \quad m_l < m \leq m_{l+1} \quad (l = 3, 4, 5, \dots).$$

Theorem 1 is now obvious since the quantities defined by (4.1) and (4.2) clearly satisfy the relations (1.4), (1.5) and (1.6).

5. Preliminary steps leading to Theorem 2. Let the assumptions of Theorem 2 be satisfied. Since they include those of Theorem 1, the existence and the properties of the four sequences $\{\omega_m\}$, $\{\eta_m\}$, $\{\rho'_m\}$, $\{\rho''_m\}$ may be taken for granted. In particular,

consider the left-hand sides of the two relations (1.6) and let n_m denote their sum; by Theorem 1

$$(5.1) \quad n_m/T(r_m) = \delta_m \rightarrow 0 \quad (m \rightarrow +\infty, T(t) = T(t, f)).$$

We define

$$(5.2) \quad \sigma_m^2 = \frac{1}{2} \min \{r_m/\rho_m', r_m/R_m', \rho_m''/r_m, R_m''/r_m, 1/\sqrt{\delta_m}\},$$

where R_m' and R_m'' are the quantities in [L, Theorem 1]; by (1.5), [L, (2.7)], and (5.1) this implies

$$(5.3) \quad \lim_{m \rightarrow \infty} \sigma_m = +\infty,$$

as well as

$$(5.4) \quad \lim_{m \rightarrow +\infty} \frac{n_m \sigma_m^2}{T(r_m)} = 0.$$

Given ε ($0 < \varepsilon < \frac{1}{2} \min \{s(0), s(\infty)\}$), we define η by the relation

$$(5.5) \quad \pi - \eta = \varepsilon/2,$$

and from now on write

$$(5.6) \quad \begin{aligned} \mathcal{S}_{0m} &= \mathcal{S}(\omega_m, \eta; \sigma_m^{-2} r_m, \sigma_m^2 r_m), \\ \mathcal{S}_{\infty m} &= \mathcal{S}(\pi + \omega_m, \eta; \sigma_m^{-2} r_m, \sigma_m^2 r_m), \\ \mathcal{A}_m &= \{z: \sigma_m^{-2} r_m < |z| \leq \sigma_m^2 r_m\}, \\ \mathcal{K}_{0m} &= \{\mathcal{A}_m - \mathcal{S}_{0m}\}, \quad \mathcal{K}_{\infty m} = \{\mathcal{A}_m - \mathcal{S}_{\infty m}\}. \end{aligned}$$

By (5.2), (5.4), (5.5) and (5.6)

$$(5.7) \quad \lim_{m \rightarrow +\infty} \frac{(n(\mathcal{S}_{0m}, 1/f) + n(\mathcal{S}_{\infty m}, f)) \sigma_m^2}{T(r_m)} = 0.$$

We propose to study the asymptotic behavior of $f(z)$ as $z \rightarrow \infty$ by values such that

$$(5.8) \quad \sigma_m^{-1} r_m < r \leq \sigma_m r_m \quad (z = r e^{i\theta}),$$

and

$$(5.9) \quad \theta \in \Gamma\left(\omega_m, \frac{s(\infty)}{2} - \varepsilon\right) = \Gamma_m.$$

Consider the fundamental representation [2, (2.6)]; for our purposes this relation may be rewritten in the form

$$(5.10) \quad \begin{aligned} \log |f(z)| &= \log \left| \prod_{a_j \in \mathcal{A}_m} \left(1 - \frac{z}{a_j}\right) \right| - \log \left| \prod_{b_j \in \mathcal{A}_m} \left(1 - \frac{z}{b_j}\right) \right| \\ &+ \log \left| \prod_{0 < |a_j| \leq \sigma_m^{-2} r_m} \left(1 - \frac{z}{a_j}\right) \right| - \log \left| \prod_{0 < |b_j| \leq \sigma_m^{-2} r_m} \left(1 - \frac{z}{b_j}\right) \right| \\ &+ \log (|c| r^q) + S(z, \sigma_m^2 r_m) \quad \left(0 < |z| = r \leq \frac{\sigma_m^2 r_m}{2}\right), \end{aligned}$$

where c ($\neq 0$) and q (an integer) are constants and the "error term" $S(z, \sigma_m^2 r_m)$ satisfies the inequality

$$|S(z, \sigma_m^2 r_m)| \leq 15 \frac{r}{\sigma_m^2 r_m} T(2\sigma_m^2 r_m).$$

By (5.2), (5.8) and [L, (2.9)], it follows that

$$(5.11) \quad |S(z, \sigma_m^2 r_m)| \leq 30\sigma_m^{-(1-\mu)} T(r) \quad (m > m_0).$$

Since for any nonrational meromorphic function f ,

$$\log r = o(T(r, f)) \quad (r \rightarrow +\infty),$$

it is obvious that (5.3) and (5.11) yield

$$(5.12) \quad |\log(|c|r^q)| + |S(z, \sigma_m^2 r_m)| = o(T(r)) \quad (r \rightarrow +\infty).$$

Let $L_m(z)$ denote the sum of the third and fourth terms in the right-hand side of (5.10). In order to estimate $L_m(z)$ we observe that if $|z|$ satisfies (5.8) and $|a| \leq \sigma_m^{-2} r_m$, then $|z|/|a| > \sigma_m$ and therefore

$$0 < -\log 2 + \log(r/|a|) < \log|1 - z/a| < \log(r/|a|) + \log 2 \quad (m > m_0),$$

which yields

$$(5.13) \quad \left| \sum_{0 < |a_j| \leq \sigma_m^{-2} r_m} \log \left| 1 - \frac{z}{a_j} \right| \right| \leq (\log 2 + 3 \log \sigma_m) n \left(\sigma_m^{-2} r_m, \frac{1}{f} \right) \\ + N \left(\sigma_m^{-2} r_m, \frac{1}{f} \right) + O(\log r) \quad (m \rightarrow \infty).$$

There is a similar formula involving the poles of $f(z)$.

By [L, (2.9)] and (5.8)

$$(5.14) \quad T(\sigma_m^{-2} r_m) < 2\sigma_m^{-\mu} T(r) \quad (m > m_0).$$

We now use (5.14) in (5.13), and in the analogous inequality for poles, and take into account [L, (2.10)], [L, (2.11)] and (5.3). This yields

$$(5.15) \quad L_m(z) = o(T(r)) \quad (m \rightarrow +\infty).$$

Denote by $\Lambda_m(z)$ the sum of the two first terms in the right-hand side of (5.10); in view of (5.12) and (5.15) we have

$$(5.16) \quad \log |f(z)| = \Lambda_m(z) + o(T(r)),$$

uniformly as $r \rightarrow \infty$ in the intervals (5.8).

The next two sections are devoted to the study of $\Lambda_m(z)$.

6. Bounds for the primary factors. Consider a zero a of $f(z)$ such that

$$(6.1) \quad a = |a|e^{i\psi}, \quad a \in \mathcal{K}_{0m}$$

and let z satisfy the conditions (5.8) and (5.9).

By (6.1) and (5.5) there exists a determination of ψ such that $\omega_m + \pi - \varepsilon/2 < \psi \leq \omega_m + \pi + \varepsilon/2$, and by (5.9) $\omega_m - s(\infty)/2 + \varepsilon < \theta \leq \omega_m + s(\infty)/2 - \varepsilon$.

Hence

$$(-s(\infty) + \varepsilon)/2 < \theta + \pi - \psi < (s(\infty) - \varepsilon)/2,$$

which, in view of the fact that $\log |1 + te^{i\phi}|$ decreases as $|\phi|$ increases from 0 to π , yields

$$(6.2) \quad \begin{aligned} \log \left| 1 - \frac{z}{a} \right| &= \log \left| 1 + \frac{r \exp [i(\theta + \pi - \psi)]}{|a|} \right| \\ &> \log \left| 1 + \frac{re^{i\lambda}}{|a|} \right| \quad (a \in \mathcal{K}_{0m}, \theta \in \Gamma_m), \end{aligned}$$

where

$$(6.3) \quad \lambda = (s(\infty) - \varepsilon)/2, \quad 0 < \lambda < \pi.$$

Similarly, if b is a pole of $f(z)$ lying in $\mathcal{K}_{\infty m}$, and if $\theta \in \Gamma_m$, then

$$(6.4) \quad \log |1 - re^{i\theta}/b| < \log |1 - re^{i\lambda}/b|.$$

The inequalities (6.2) and (6.4), and our definition of $\Lambda_m(z)$, yield

$$(6.5) \quad \begin{aligned} \Lambda_m(z) &> \log \left| \prod_{a_j \in \mathcal{A}_m} \left(1 + \frac{re^{i\lambda}}{|a_j|} \right) \right| - \log \left| \prod_{b_j \in \mathcal{B}_m} \left(1 - \frac{re^{i\lambda}}{|b_j|} \right) \right| \\ &\quad - \log \left| \prod_{a_j \in \mathcal{S}_{0m}} \left(1 + \frac{re^{i\lambda}}{|a_j|} \right) \right| + \log \left| \prod_{b_j \in \mathcal{S}_{\infty m}} \left(1 - \frac{re^{i\lambda}}{|b_j|} \right) \right| \\ &\quad + \log \left| \prod_{a_j \in \mathcal{S}_{0m}} \left(1 - \frac{z}{a_j} \right) \right| - \log \left| \prod_{b_j \in \mathcal{S}_{\infty m}} \left(1 - \frac{z}{b_j} \right) \right| \quad (\theta \in \Gamma_m). \end{aligned}$$

If $a \in \mathcal{A}_m$ and r satisfies (5.8), we have $\sin \lambda \leq |1 + re^{i\lambda}/a| < 2\sigma_m^3$, and hence, by (5.4) and [L, (2.9)],

$$(6.6) \quad \begin{aligned} \sum_{a_j \in \mathcal{S}_{0m}} \left| \log \left| 1 + \frac{re^{i\lambda}}{|a_j|} \right| \right| + \sum_{b_j \in \mathcal{S}_{\infty m}} \left| \log \left| 1 - \frac{re^{i\lambda}}{|b_j|} \right| \right| \\ \leq n_m(\log 2 + 3 \log \sigma_m) = o(T(r)) \quad (m > m_0, r \rightarrow +\infty). \end{aligned}$$

The two last terms of (6.5) are estimated by the following straight-forward application of the lemma of Boutroux-Cartan: if $z \in \mathcal{A}_m$ and if z avoids finitely many disks with sum of diameters equal to $\sigma_m^{-2}r_m/2$, we have

$$\begin{aligned} \prod_{a_j \in \mathcal{S}_{0m}} |z - a_j| &\geq \left(\frac{\sigma_m^{-2}r_m}{8e} \right)^n \quad (n = n(\mathcal{S}_{0m}, 1/f)), \\ (1 + \sigma_m^4)^n &\geq \prod_{a_j \in \mathcal{S}_{0m}} \left| 1 - \frac{z}{a_j} \right| \geq (8e\sigma_m^4)^{-n}. \end{aligned}$$

The same bounds hold for the polynomial formed with the poles b_j ($\in \mathcal{S}_{\infty m}$). Hence, the arguments used in the proof of (6.6), yield

$$(6.7) \quad \left| \log \left| \prod \left(1 - \frac{z}{a_j} \right) \right| \right| + \left| \log \left| \prod \left(1 - \frac{z}{b_j} \right) \right| \right| \\ \leq n_m(\log(8e) + 4 \log \sigma_m) = o(T(r)) \quad (r \rightarrow \infty),$$

provided r (confined to $\mathcal{A}_m$) avoids a set $\mathcal{E}_m$, of measure not greater than $\sigma_m^{-2} r_m$.

7. Proof of Theorem 2. Let $I_1(r)$ denote the first term in the right-hand side of (6.5). The elementary identity

$$(7.1) \quad \log \left| 1 + \frac{z}{a} \right| = \Re e \left\{ z \int_{|a|}^{+\infty} \frac{dt}{t(z+t)} \right\} \quad (a \neq 0),$$

valid if z is not real and negative, shows that

$$(7.2) \quad I_1(r) = \Re e \left\{ z \int_{\sigma_m^{-2} r_m}^{\sigma_m^2 r_m} \frac{n(t, 1/f) - n(\sigma_m^{-2} r_m, 1/f)}{t(z+t)} dt \right. \\ \left. + z \int_{\sigma_m^2 r_m}^{+\infty} \frac{n(\sigma_m^2 r_m, 1/f) - n(\sigma_m^{-2} r_m, 1/f)}{t(z+t)} dt \right\} \quad (z = re^{i\lambda}),$$

and using again (7.1)

$$(7.3) \quad I_1(r) = \Re e \left\{ z \int_{\sigma_m^{-2} r_m}^{\sigma_m^2 r_m} \frac{n(t, 1/f)}{t(z+t)} dt \right\} + n(\sigma_m^2 r_m, 1/f) \log \left| 1 + \frac{z}{\sigma_m^2 r_m} \right| \\ - n(\sigma_m^{-2} r_m, 1/f) \log \left| 1 + \frac{z}{\sigma_m^{-2} r_m} \right| \quad (z = re^{i\lambda}).$$

Now $\log |1 + z/\sigma_m^2 r_m| = O(r/\sigma_m^2 r_m)$, and

$$\log \left| 1 + \frac{z}{\sigma_m^{-2} r_m} \right| = \log \left(\frac{r \sigma_m^2}{r_m} \right) + o(1),$$

uniformly as $r \rightarrow +\infty$ in the intervals $(\sigma_m^{-1} r_m, \sigma_m r_m]$. Hence, obvious estimates using [L, (2.9)] and [L, (2.11)] show that (7.3) reduces to

$$(7.4) \quad I_1(r) = \Re e \left\{ z \int_{\sigma_m^{-2} r_m}^{\sigma_m^2 r_m} \frac{n(t, 1/f)}{t(z+t)} dt \right\} + o(T(r)) \\ (z = re^{i\lambda}, r \rightarrow +\infty, \sigma_m^{-1} r_m < r \leq \sigma_m r_m).$$

Rewriting (7.4) in the form

$$\frac{I_1(r)}{T(r)} = \Re e \left\{ z \int_{\sigma_m^{-2} r_m}^{\sigma_m^2 r_m} \frac{n(t, 1/f)}{T(t)} \frac{T(t)}{T(r)} \frac{dt}{t(z+t)} \right\} + o(1),$$

we obtain, in view of [L, (2.9)] and [L, (2.11)],

$$(7.5) \quad \frac{I_1(r)}{T(r)} = \Re e \left\{ z \mu u \int_{\sigma_m^{-2} r_m}^{\sigma_m^2 r_m} \left(\frac{t}{r} \right)^\mu \frac{dt}{t(z+t)} \right\} + o(1) \\ = \Re e \left\{ \mu u e^{i\lambda} \int_0^\infty \frac{x^{\mu-1} dx}{x + e^{i\lambda}} \right\} + o(1),$$

as $r \rightarrow +\infty$ in the intervals $(\sigma_m^{-1}r_m, \sigma_m r_m]$. The value of the last integral in (7.5) is well known to be $\pi e^{i\lambda(\mu-1)}/\sin \pi\mu$, and hence we are finally led to

$$(7.6) \quad \frac{I_1(r)}{T(r)} = \frac{1}{T(r)} \log \left| \prod_{a_j \in \mathcal{A}_m} \left(1 + \frac{r e^{i\lambda}}{|a_j|} \right) \right| = \frac{\pi\mu u}{\sin \pi\mu} \cos \lambda\mu + o(1).$$

The same method yields

$$(7.7) \quad \frac{1}{T(r)} \log \left| \prod_{b_j \in \mathcal{B}_m} \left(1 - \frac{r e^{i\lambda}}{|b_j|} \right) \right| = \frac{\pi\mu v}{\sin \pi\mu} \cos (\pi - \lambda)\mu + o(1).$$

Combining (5.16), (6.5), (6.6), (6.7), (7.6) and (7.7) we obtain, uniformly in z ,

$$(7.8) \quad \log |f(z)| \geq \frac{\pi\mu T(r)}{\sin \pi\mu} \{u \cos (\lambda\mu) - v \cos (\pi - \lambda)\mu\} + o(T(r)),$$

provided

- (i) $|z| = r \rightarrow +\infty$ in the intervals $(r_m \sigma_m^{-1}, r_m \sigma_m]$;
- (ii) r avoids the exceptional sets $\mathcal{E}_m$ (meas $\mathcal{E}_m \leq \sigma_m^{-2} r_m$);
- (iii) $\arg z = \theta \in \Gamma\left(\omega_m, \frac{s(\infty)}{2} - \varepsilon\right)$.

If we choose any K such that

$$(7.9) \quad 0 < K < \frac{\pi\mu}{\sin \pi\mu} \left(u \cos \left\{ \frac{s(\infty)\mu}{2} - \frac{\varepsilon\mu}{2} \right\} - v \cos \left\{ \left(\pi - \frac{s(\infty)}{2} \right) \mu + \frac{\varepsilon\mu}{2} \right\} \right) = \tilde{K},$$

we see that (7.8) and (6.3) imply the first of the inequalities (1.7). We must still verify that $\tilde{K} > 0$ since otherwise it will be impossible to find a K satisfying (7.9). The relations [L, (2.4)], [L, (2.5)] and [L, (2.6)] yield an explicit value of $\tilde{K}$:

$$\tilde{K} = \frac{\pi\mu}{\sin \pi\mu} \left\{ u \sin \left(\frac{s(\infty)\mu}{2} \right) + v \sin \left(\frac{s(o)\mu}{2} \right) \right\} \sin \frac{\varepsilon\mu}{2} > 0.$$

The second inequality (1.7) is obtained by considering $1/f$ instead of f . Our proof of Theorem 2 is now complete.

8. Proof of Theorem 3. Assume that the Theorem is false. Then there exists a meromorphic function $F(z)$ of lower order μ ($0 < \mu < 1$) having at least two finite, distinct, deficient values τ_1, τ_2 and such that $f(z) = F'(z)$ satisfies the conditions of Theorem 1.

By the elements of Nevanlinna's theory

$$(8.1) \quad m(r, f/F) + m(r, f/(F - \tau_1)) + m(r, f/(F - \tau_2)) = o(T(r, F))$$

$(r \notin \mathcal{E}, r \rightarrow +\infty),$

where $\mathcal{E}$ is an exceptional set of finite measure. It is well known that this relation implies

$$(8.2) \quad T(r, f) \leq 2T(r, F)(1 + o(1)) \quad (r \notin \mathcal{E}, r \rightarrow +\infty),$$

and also (since the relation [L, (9.3)] is valid with g replaced by F),

$$(8.3) \quad N(r, 1/f) + m(r, 1/(F - \tau_1)) + m(r, 1/(F - \tau_2)) \leq T(r, f) + o(T(r, F)) \\ (r \notin \mathcal{E}, r \rightarrow +\infty).$$

From the definition of deficient value, we deduce

$$(8.4) \quad m(r, 1/(F - \tau_k)) > \frac{1}{2} \delta(\tau_k, F) T(r, F) \quad (r > r_0; k = 1, 2),$$

and hence, in view of (8.2) and (8.3), there exist two constants κ_1, κ_2 such that

$$(8.5) \quad 0 < \kappa_1 < T(r, f)/T(r, F) < \kappa_2 < +\infty \quad (r \notin \mathcal{E}, r > r_0).$$

Let J be any measurable subset of C such that $\text{meas } \{J\} = 4\epsilon > 0$; then, by a lemma of Edrei and Fuchs [7, p. 322, Lemma III],

$$(8.6) \quad \frac{1}{2\pi} \int_J \log^+ \left| \frac{1}{F - \tau_k} \right| d\theta = m\left(r, \frac{1}{F - \tau_k}; J\right) \leq A_0 T(2r, F) \epsilon \left(1 + \log^+ \frac{1}{\epsilon}\right) \\ (r > r_0; k = 1, 2),$$

where A_0 is an absolute constant.

From now on r will be restricted to the intervals $(r_m, 2r_m]$, and $\{r_m\}$ is the sequence of Pólya peaks (of $T(r, f)$) which appears in Theorems 1 and 2. By [L, (2.9)], (8.5) and the fact that the characteristic functions are increasing,

$$(8.7) \quad T(2r, F) < K_0 T(r, F) \quad (r_m < r \leq 2r_m, r \notin \mathcal{E}, m > m_0);$$

the constant K_0 depends only on κ_1, κ_2 and μ .

Using (8.7) in (8.6) we obtain

$$(8.8) \quad m\left(r, \frac{1}{F - \tau_k}; J\right) \leq A_0 K_0 T(r, F) \epsilon \left(1 + \log^+ \frac{1}{\epsilon}\right) \\ (r_m < r \leq 2r_m, r \notin \mathcal{E}, m > m_0; k = 1, 2),$$

and choose ϵ ($0 < \epsilon < \frac{1}{2} \min(s(0), s(\infty))$) so small that the right-hand side of (8.8) is less than

$$\frac{1}{4} \min \{\delta(\tau_1, F), \delta(\tau_2, F)\} T(r, F).$$

We use this value of ϵ in Theorem 2 and select a sequence $\{\tilde{r}_m\}$ such that

$$r_m < \tilde{r}_m \leq 2r_m, \quad \tilde{r}_m \notin \mathcal{E}, \quad \tilde{r}_m \notin \mathcal{E}_m \quad (m > m_0).$$

This is certainly possible because $\mathcal{E}$ is of finite measure and

$$\text{meas } \mathcal{E}_m \leq \sigma_m^{-2} r_m = o(r_m) \quad (m \rightarrow \infty).$$

The set

$$J_m = C - \{\Gamma(\omega_m, s(\infty)/2 - \epsilon) \cup \Gamma(\pi + \omega_m, s(0)/2 - \epsilon)\}$$

is of measure 4ϵ and hence (8.8) and our choice of ϵ and $\tilde{r}_m$ imply

$$(8.9) \quad m(\tilde{r}_m, 1/(F - \tau_k); J_m) < \frac{1}{4} \delta(\tau_k, F) T(\tilde{r}_m, F) \quad (k = 1, 2).$$

Now (8.1), the first relation (1.7) and the elementary inequality

$$\log^+ |1/(F - \tau_k)| \leq \log^+ |f/(F - \tau_k)| + \log^+ |1/f|,$$

yield

$$(8.10) \quad m(\tilde{r}_m, 1/(F - \tau_k); \Gamma(\omega_m, s(0)/2 - \varepsilon)) = o(T(\tilde{r}_m, F)) \quad (m \rightarrow \infty, k = 1, 2).$$

If we consider the inequalities (8.4) with $r = \tilde{r}_m$, and compare them with (8.9) and (8.10), we see that for m large enough, there will exist points

$$z_{1m} = \tilde{r}_m \exp(i\theta_{1m}), \quad z_{2m} = \tilde{r}_m \exp(i\theta_{2m}),$$

such that $\theta_{1m}, \theta_{2m} \in \Gamma(\omega_m + \pi, s(0)/2 - \varepsilon)$,

$$(8.11) \quad |F(z_{1m}) - \tau_1| < \frac{1}{3}|\tau_2 - \tau_1|, \quad |F(z_{2m}) - \tau_2| < \frac{1}{3}|\tau_2 - \tau_1|.$$

Let $\mathcal{C}_m$ denote the subinterval of $\Gamma(\omega_m + \pi, s(0)/2 - \varepsilon)$ having end points θ_{1m}, θ_{2m} . Then, the obvious relation

$$|F(z_{1m}) - F(z_{2m})| = \left| \int_{\mathcal{C}_m} f(\tilde{r}_m e^{i\theta}) \tilde{r}_m e^{i\theta} d\theta \right|,$$

the second relation (1.7), and the fact that $\log \tilde{r}_m = o(T(\tilde{r}_m, F))$, imply

$$(8.12) \quad |F(z_{1m}) - F(z_{2m})| < \frac{1}{3}(\tau_2 - \tau_1) \quad (m > m_0).$$

The inequalities (8.11) and (8.12) are clearly incompatible. This contradiction shows that $F(z)$ cannot have the finite, distinct, deficient values τ_1, τ_2 , and hence proves Theorem 3.

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